**Chapter 6 Algebra and Analysis**

**6.1: A little Algebra**

**Problem 6.1.1:-** Show that if n is a positive integer then n3-n is divisible by 3.

**Soln:** Given n3-n

=n (n2-1)

=n (n+1) (n-1)

These three consecutive number of positive integer, out of these three one number must divisible by 3.Hence n3-n is divisible by 3.

**Problem 6.1.2:-** Show that if n is a positive integer then n5-n always divisible by 5.

**Soln:** Given n5-n

=n (n4-1)

=n (n2-1) (n2+1)

=n (n-1)(n+1)(n2+1)

If n is integer ending with 0,1,4,5,6,9 then one of n (n+1)(n-1) is divisible by 5.If n is integer ending with 2,3,7,8 then n2+1 is divisible by 5.Hence for every n,n5-n is divisible by 5.

**Problem 6.1.3:**-Show that if n is a positive integer then n7-n is divisible by 7.

**Soln:** Given n7-n

=n (n6-1)

=n (n3-1)(n3+1)

=n (n-1)(n2+n+1)(n-1)(n2-n+1)

For n=0, 0/7=0, which is divisible by 7.

For n=1, 0/7=0, which is divisible by 7.

For n=2, n2+n+1/7=22+2+1/7=1, which is divisible by 7.

For n=3, n2-n+1/7=32-3+1/7=1, which is divisible by 7.

For n=4, n2+n+1/7=42+4+1/7=3, which is divisible by 7.

For n=5, n2-n+1/7=52-5+1/7=3, which is divisible by 7.

For n=6, n+1/7=6+1/7=1 which is divisible by 7.

For n=8, n-1/7=8-1/7=1 which is divisible by 7.

For n=9, n2+n+1/7=92+9+1/7=13 which is divisible by 7.

Hence for all n, n7-n is divisible by 7.

**Problem 6.1.4:-** Verify combinational identify.

kcm+ kcm+1= k+1cm+1

**Soln:** L.H.S. = kcm+ kcm+1(ncr=n!/(n-r)!r!)

=k!/(k-m)m! + k!/(k-m-1)!(m+1)!

= k!(m+1)/(k-m)(m+1)! + k! (k-m)/(k-m)!(m+1)!

=k!(m+1+k-m)/(k-m)!(m+1)!

=k!(k+1)/(k-m)!(m+1)!

=(k+1)!/(k-m)!(m+1)!

=k+1Cm+1

Therefore L.H.S. =R.H.S.

Hence Proved

**Problem 6.1.6:-** Which is greater α=(1+0.00001)1000000 or 2?

**Soln:** Here given,

α=(1+1/1000000)1000000

We put k=1000000, then α=(1+1/k)k which is **e** whose value is 2.718. Hence, α is greater than 2.

**Problem 6.1.7:-** Which is greater? (1000)1000 or (1001)999.

**Soln:**- We have (1001)999

=(1000+1)999

[Formula(x+y)n=ncoxnyo+nc1  xn-1y1+nc2 xn-2y2+….+ncnx0yn]

=999co1000999+999c1  1000999-111+999c2 1000999-212+….+999c999100001999

=1. 1000999+999.1000998+1……+1 < 1.10001000+1000999+….+1000900

(1000 terms)

=1000999 \*1000

=10001000

**Therefore**, [1000+1]999<10001000

[1001]999<10001000

**Problem 6.1.8:-** Assume that k is a positive integer.Calculate

1/1.2 + 1/2.3 +…………..+1/(k-1)k +1\k(k+1)

**Soln:** Here ,

1/k(k+1) = 1/k – 1/(k+1)

Let, Sk = [1-1\2]+[1/2-1/3]+[1/3-1/4]+….+[1/(k-1)-1/k]+[1/k-1/(k+1)]

= 1 - ~~1/2~~ + ~~1/2~~ - ~~1/3~~ + ~~1/3~~ - ~~1/4~~ + ~~1/4~~ +……….+ ~~1/k-1~~ - ~~1/k~~ + ~~1/k~~ - 1/(k-1)

= 1 - 1/k+1

= k/k+1

**Problem 6.1.10:-** Calculate the sum (1.2)+(2.3)+(3.4)+…….+n(n+1)

**Soln:** Let Sn =1.2+2.3+3.4+……..+n(n+1)

=2(1+3) + 3(2+4) + 4(3+5) + n [(n-1)+(n+1)]

=2[(1.2+2.3+3.4+…..+n (n+1)] - 1.2 – n(n+1)

We subtracted 1.2 and n(n+1) because it uses only one time.

2.4 + 3.6 + 4.8 + ….+ n.2n = Sn – 1.2 – n(n+1)

or, 22 . 2 + 32 . 2 + 42 . 2 +n2 .2 = Sn – 2 – n(n+1)

or, 2{22 + 32 + 42 +n2} = Sn – 2 – n(n+1)

or, 2 {12 + 22 + 32 + 42 +n2} = Sn – ~~2~~ – n(n+1) + ~~2~~

or, 2 n (n+1)(2n+1)/6 = Sn – n(n+1)

or, Sn = n (n+1)(2n+1)/3 + n(n+1)

or, Sn = [n (n+1)(2n+1) + 3n(n+1)]/3

or, Sn = [n (n+1) (2n+1+3)] / 3

or, Sn = [2n(n+1)(n+2)]/3

**Challenging Problem 6.1.11:-** Calculate the sum 1.2.3 + 2.3.4 + …………+ n (n+1) (n+2)

**Soln:** Given, 1.2.3 + 2.3.4 + …………+ n (n+1) (n+2)

So, it’s nth term is tn = n (n+1) (n+2)

Sn=

=

=

=

= +3 +

= ( + +

= + + n (n+1)

=n(n+1) [ + +1 ]

=

Therefore, Sn=

**Problem 6.2.1:-** If a and b are positive real numbers then show that ab ≤

**Soln:** Given, ab ≤

or, 2ab ≤ a2 + b2

or, a2 - 2ab + b2 ≥ 0

or, 0≤(a-b)2

The square of any value always positive, so the expression 0≤(a-b)2 is always true. Hence the inequality ab ≤ hold.

**Problem 6.2.3:-** Prove that, 2 < 1/log2π + 1/log5π

Formula=logba = ln(a)/ln(b)

**Soln:** Here given 2 < 1/log2π + 1/log5π

or, 2 < 1/ln(π)/ln(2) + 1/ln(π)/ln(5)

or, 2 < ln2/lnπ + ln5/lnπ

or, 2 < ln2 + ln5/lnπ

or, 2 lnπ < ln(2.5)

or, lnπ2 < ln10

or, π2 < 10

Here , π=3.14 whose squaring value is always less than 10.

Hence the given inequality always hold.

**Problem 6.2.5:-** Show that, |cosx + sinx|≤√2 with equality only if sin2x=1.

(Formula: |α|=

**Soln: |**sinx + cosx**|**=

=

=

Clearly greater value of sin2x is 1.

Therefore, |cosx + sinx|≤

**Challenging Problem 6.2.6:-** Show that |cosx-sinx|≤

**Soln:**  |cosx-sinx|=

=

=

Clearly least value of sin2x is -1.

Therefore, |cosx-sinx|≤

**Problem 6.2.7:-** Which is greater sin(cosx) or cos(sinx)?

**Soln:** Let cos()

=cos.cos(cosx) -sin.sin(cosx)

= -sin(cosx)

Therefore, cosx(sinx)- sin(cosx)

= cosx(sinx) + cosx(cosx+)

Again,

cosx+cosy= 2cos(x+y/2) – cos(x-y/2)

Then from equn (i) and (ii),

Therefore ,

cos(sinx) – sin(cosx) = cos(sinx) + cos(cos+)

= 2 cos

Now,

0<||≤||+||

≤

< 1.5<

Again,

0<||≤||+||

≤

< 1.5<

Hence from equn (iii)

cos(sinx) – sin(cosx) < 2 cos().cos()

≤ 2.1.1

Therefore,

Cos(sinx) – sin (cosx) < (+ve value)

Hence, difference of two value is +ve, so cos(sinx) is greater than sin(cosx)

**6.3 Trigonometry and Related Ideas**

**Problem 6.3.1:-**

Suppose that α is an angle and tan() is rational.Verify that sinα and cosα are both rational.

**Soln:** We know that,

1+tan2 **=** sec2

**=** 1/cos2

Since tan() is rational.Here left hand side is rational so by equality right hand side is also rational. Hence cos2 α/2 is also rational.

Again, cosα = cos2α/2 – sin2 α/2

= 2 cos2α/2 – 1

Since cos2α/2 is rational, so right hand side is rational. Again by equality, we can say cosα is also rational.

Again, tanα =

= (2sinα/2.cosα/2)/ (cos2α/2 – sin2 α/2)

= 2 tan()/ 1- tan2

We know each component of right hand side is rational,so by equality we can say tanα is rational. But since cosα is rational.We say that sinα is also rational.

**Problem 6.3.3:-** If O is positive, acute angle measure in radians , then show that tanθ > θ .

**Soln:** Let, OB= 1 unit ( radius of circle)

Here, triangle AOB is similar with triangle COD

From triangle AOB

cos θ = OB/OA = 1/OA

OA= 1/ cos θ = hypotenuse of triangle AOB

Therefore, tanθ=AB/OB = sinθ/cosθ = height of triangle AOB

Also, the height of triangle is greater than the length of the arc of the circle

that is subtended.

AB > BC (θ=l/r= BC)

tanθ > θ (BC = θ)

Therefore, sinθ/cosθ > θ

**Problem 6.3.5:-** Suppose that α be any angle. Explain why.

cos cos . cos =

**Soln:** L.H.S.=

=

=

=

=

= R.H.S. Proved

**Problem:- Prove that if 0≤a,b,c,d≤1, then (1-a)(1-b)(1-c)(1-d)≥1-a-b-c-d**

**Soln:** Proof,

L.H.S.= (1-a)(1-b)(1-c)(1-d)

= (1-b-a+ab)(a-d-c+cd)

=(1-d-c+cd-b+bd+bc-bcd-a+ad+ac-acd+ab-abd-abc+abcd)

= (1-a-b-c+ab+ac+ad+bc+bd+cd-abc-abd-acd-bcd+abcd)

=[1-a-b-c-d+ab(1-c)+bc(1-d)+cd(1-a)+ad(1-b)+ac+bd+abcd]

Therefore , (1-a)(1-b)(1-c)(1-d)≥(1-a-b-c-d)

**Problem:- Explain why 1110-1 is divisible by 100.**

**Soln:** Given, 1110-1

= (10+1)10-1

= [10C0(10)10.10+10C1(10)9.11+10C2(10)8 12+…………+10C9 .101 19+10C10 100 .110 ]

=[1010 +1010 +45.108+………….+102+1-1]

=10 [108+108+45.106+……….+1]

Hence 1110-1 is divisible by 100.